Translational groups as generators of gauge transformations

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We examine the gauge generating nature of the translational subgroup of Wigner's little group for the case of massless tensor gauge theories and show that the gauge transformations generated by the translational group are only a subset of the complete set of gauge transformations. We also show that, just as in the case of topologically massive gauge theories, translational groups act as generators of gauge transformations in gauge theories obtained by extending massive gauge noninvariant theories by a Stückelberg mechanism. The representations of the translational groups that generate gauge transformations in such Stückelberg extended theories can be obtained by the method of dimensional descent. We illustrate these results with the examples of Stückelberg extended first class versions of Proca, Einstein-Pauli-Fierz, and massive Kalb-Ramond theories in $3+1$ dimensions. A detailed analysis of the partial gauge generation in massive and massless second rank symmetric gauge theories is provided. The gauge transformations generated by the translational group in two-form gauge theories are shown to explicitly manifest the reducibility of gauge transformations in these theories.

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I. INTRODUCTION

Wigner's little group is quite familiar to physicists mainly because of its role in the classification of elementary particles. Wigner introduced the concept of the little group in a seminal paper [1] published in 1939 and showed how particles can be classified on the basis of their spin or helicity quantum numbers using the little group. The little group also relates the internal symmetries of massive and massless particles [2]. A comparatively lesser known facet of the little group is its role as a generator of gauge transformations in various Abelian gauge theories. This aspect of the little group was first noticed in the context of free Maxwell theory [3–5] and linearized Einstein gravity [6]. Recently, it was shown that the little group for massless particles acts as a generator of gauge transformations in the case of other gauge theories as well [7]. For example, the defining representation of this little group is shown to generate gauge transformations also in the $(3+1)$-dimensional Kalb-Ramond (KR) theory, which is a massless two-form gauge theory [8,9]. To be precise, it is the translational subgroup[1] $T(2)$ of Wigner's little group for massless particles that generates gauge transformations in these theories. On the other hand, in $(3+1)$-dimensional $B\wedge F$ theory, which is a topologically massive gauge theory, one needs to go beyond $T(2)$, and it is a particular representation of the translational group $T(3)$ that generates the gauge transformations in this theory. However, as shown in [10], one can easily see that $T(3)$ is a subgroup of Wigner's little group for a massless particle in $4+1$ dimensions, which generates gauge transformations in massless theories living in this higher dimensional space-time. It is further shown in [10] that one can systematically derive the representation of $T(3)$ that acts as gauge generator in $(3+1)$-dimensional $B\wedge F$ theory from the gauge transformation properties of the free Maxwell and KR theories in $4+1$ dimensions using a method called "dimensional descent." Similarly, dimensional descent can also be employed to obtain the gauge generating representations of $T(1)$ for topologically massive Maxwell-Chern-Simons and linearized Einstein-Chern-Simons gauge theories [11–13] in $2+1$ dimensions by starting, respectively, from Maxwell and linearized gravity theories in $3+1$ dimensions [10,14].

Against this background, the purposes of the present study are the following. First, we make a closer analysis of the gauge generation by Wigner's little group in massless tensor gauge theories, namely, linearized gravity and Kalb-Ramond theories, and unravel certain subtle points that went unnoticed in earlier studies. We show that the translational group $T(2)$ generates only a subset of the full range of gauge transformations in these theories. Furthermore, in the case of KR and $B\wedge F$ theories, the generators of gauge transformations are not all independent, and such theories are known as "reducible gauge systems" [15,16]. Our analysis shows that gauge generation by the translational group $T(2)$ in a reducible gauge theory manifestly exhibits the reducibility of the gauge transformations (Sec. III).

Second, one should note that, apart from the usual massless gauge theories and topologically massive gauge theories, there exist gauge theories that can be obtained by converting second class constrained systems (in the language of Dirac's theory of constraint dynamics) to first class (gauge) systems using the generalized canonical prescription of Batalin, Fradkin, and Tyutin [17]. By such a prescription one can obtain from the massive gauge noninvariant theories their Stückelberg extended versions, which are massive as well as gauge invariant [18]. Now, one may wonder if translational groups act as gauge generators in such massive gauge theories as well. If so, what would be the representations of these groups that generate such gauge transformations? In the present study, we delve into these questions and show that the same representation of the translational group $T(3)$ that

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[1]A group of translations in n-dimensional space is denoted by $T(n)$.

generates gauge transformation in the topologically massive $B\wedge F$ theory also generates gauge transformation in the Stückelberg extended versions of the Proca, Einstein-Pauli-Fierz (EPF), and massive KR theories (Sec. IV).

Finally, it will be shown that, just as in topologically massive gauge theories, dimensional descent can also be used to obtain the polarization vectors (or tensors) and momentum vectors of the Stückelberg models and the gauge generating representation of the translational group in such models, by starting from appropriate theories in one higher space-time dimension (Sec. V).

Notation. We use μ,ν, etc., for denoting indices in $(3+1)$-dimensional space-time. The letters i,j, etc., are used for $(4+1)$-dimensional space-time except in Sec. IV, where they represent spatial components of $(3+1)$-dimensional vectors and tensors. The metric used is mostly negative. We denote polarization vectors by ε_μ, polarization tensors of two-form theories by $\varepsilon_{\mu\nu}$, and those of symmetric second rank tensor fields by $\chi_{\mu\nu}$.

II. WIGNER'S LITTLE GROUP AS A GENERATOR OF GAUGE TRANSFORMATIONS IN VARIOUS THEORIES

Historically, the gauge generating property of the little group was first studied in the context of free Maxwell theory [3–5], where it was shown that the action of the little group on the polarization vector of Maxwell photons amounts to a gauge transformation in momentum space. As is well known, free Maxwell theory is described by the Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \tag{1}$$

which is invariant under the gauge transformation $A_\mu(x)\rightarrow A_\mu(x)+\partial_\mu\tilde{f}(x)$, where $\tilde{f}(x)$ is an arbitrary scalar function. The Lagrangian (1) leads to the equation of motion $\partial_\mu F^{\mu\nu}=0$. Denoting the polarization vector of a photon by $\varepsilon^\mu(k)$, a solution of this equation can be written as

$$A^\mu(x)=\varepsilon^\mu(k)e^{ik\cdot x}, \tag{2}$$

where only a single mode is considered and the positive frequency part is suppressed for simplicity. In terms of the polarization vector $\varepsilon^\mu(k)$, the gauge transformation and the equation of motion for Maxwell theory are expressed, respectively, as follows;

$$\varepsilon_\mu(k)\rightarrow\varepsilon'_\mu(k)=\varepsilon_\mu(k)+if(k)k_\mu, \tag{3}$$

$$k^2\varepsilon^\mu-k^\mu k_\nu\varepsilon^\nu=0, \tag{4}$$

where $\tilde{f}(x)$ has been written as $\tilde{f}(x)=f(k)e^{ik\cdot x}$. The massive excitations corresponding to $k^2\neq 0$ lead to the solution $\varepsilon^\mu\propto k^\mu$, which can therefore be gauged away by a suitable choice of $f(k)$ in Eq. (3). For massless excitations ($k^2=0$), the Lorentz condition $k_\mu\varepsilon^\mu=0$ follows immediately from Eq. (4). For a photon of energy ω propagating in the z direction, the four-momentum can be written as $k^\mu=(\omega,0,0,\omega)^T$. It then follows from Eq. (4) that the corresponding polarization tensor $\varepsilon^\mu(k)$ takes the form $(\varepsilon^0,\varepsilon^1,\varepsilon^2,\varepsilon^0)$, which can be reduced to the maximally reduced form[2]

$$\varepsilon^\mu(k)=(0,\varepsilon^1,\varepsilon^2,0)^T \tag{5}$$

by the gauge transformation (3) with $f(k)=i\varepsilon^0/\omega$. Note that the maximally reduced form (5) of ε^μ displays just the two physical degrees of freedom ε^1 and ε^2.

We now digress briefly to recapitulate the essential aspects of Wigner's little group. Wigner's little group $\mathcal{W}$ is defined as the subgroup of the homogeneous Lorentz group L that preserves the energy-momentum vector of a particle:

$$\mathcal{W}^\mu{}_\nu k^\nu=k^\mu, \tag{6}$$

where k^μ is an arbitrary, but fixed, vector on the mass shell $\mathcal{M}_{m^2}=\{k^\mu|k^2=m^2\}$. $\mathcal{M}_{m^2}$ is acted on transitively by the Lorentz group L. The little group $W(k)$ is the stability subgroup of L so that $\mathcal{M}_{m^2}$ can be identified as a homogeneous coset space $L/\mathcal{W}$. It is obvious that, in $3+1$ dimensions, the little group for a massive particle is the rotational group $SO(3)$. On the other hand, for a massless particle, the little group is isomorphic to the Euclidean group $E(2)$, which is a semidirect product of $SO(2)$ and $T(2)$—the group of translations in the two-dimensional plane [5]. The explicit representation of Wigner's little group $\mathcal{W}_4$ that preserves the four-momentum $k^\mu=(\omega,0,0,\omega)^T$ of a photon of energy ω moving in the z direction is given by [3]

$$\mathcal{W}_4(p,q;\phi)=\begin{pmatrix} 1+\frac{p^2+q^2}{2} & p\cos\phi-q\sin\phi & p\sin\phi+q\cos\phi & -\frac{p^2+q^2}{2} \\ p & \cos\phi & \sin\phi & -p \\ q & -\sin\phi & \cos\phi & -q \\ \frac{p^2+q^2}{2} & p\cos\phi-q\sin\phi & p\sin\phi+q\cos\phi & 1-\frac{p^2+q^2}{2} \end{pmatrix}. \tag{7}$$

[2]This procedure of obtaining the maximally reduced polarization vectors or tensors of various theories by choosing a plane wave solution for the corresponding equation of motion will henceforth be referred to as the ''plane wave method.''

Here p and q are real parameters. This little group can be written as $W_4(p,q;\phi)=W(p,q)R(\phi)$, where

$$W(p,q)\equiv\mathcal{W}_4(p,q;0)=\begin{pmatrix} 1+\frac{p^2+q^2}{2} & p & q & -\frac{p^2+q^2}{2} \\ p & 1 & 0 & -p \\ q & 0 & 1 & -q \\ \frac{p^2+q^2}{2} & p & q & 1-\frac{p^2+q^2}{2} \end{pmatrix} \quad (8)$$

is a particular representation of the translational subgroup $T(2)$ of the little group and $R(\phi)$ represents a $SO(2)$ rotation about the z axis. Note that the representation $W(p,q)$ satisfies the relation $W(p,q)W(\bar{p},\bar{q})=W(p+\bar{p},q+\bar{q})$.

Under the action of the translational group $T(2)$ in Eq. (8), the maximally reduced polarization vector (5) of Maxwell theory transforms as follows:

$$\varepsilon^{\mu}\rightarrow\varepsilon'^{\mu}=W^{\mu}{}_{\nu}(p,q)\varepsilon^{\nu}=\varepsilon^{\mu}+\left(\frac{p\varepsilon^1+q\varepsilon^2}{\omega}\right)k^{\mu}. \quad (9)$$

Clearly, this can be identified as a gauge transformation of the form of (3) by choosing $f(k)=(p\varepsilon^1+q\varepsilon^2)/i\omega$, thus displaying the gauge generating property of Wigner's little group for massless particles in free Maxwell theory. Conversely, any general gauge transformation (in momentum space) in Maxwell theory can be viewed as resulting from the action of the translational group $W(p,q)$ on the polarization vector of the theory.

The same translational group $T(2)$ in the representation (8) generates gauge transformations in $(3+1)$-dimensional linearized gravity [14] and Kalb-Ramond theory [8]. However, as we will show in the next section, the transformation generated by $T(2)$ is only a subset of the whole spectrum of gauge transformations in these massless tensor gauge theories. On the other hand, in the case of the topologically massive $B\wedge F$ gauge theory [19], the translational group (8) fails to act as a gauge generator. The generator of gauge transformations in $B\wedge F$ theory is shown in [8] to be the translational group $T(3)$ in the representation

$$D(p,q,r)=\begin{pmatrix} 1 & p & q & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (10)$$

(where p,q,r are real parameters). Another representation of $T(3)$, inherited from the defining representation of Wigner's little group for massless particles in $4+1$ dimensions, generates gauge transformations in $(4+1)$-dimensional Maxwell theory. The close relationship between these different representations of $T(3)$ is analyzed in detail in [10] using the method called dimensional descent, which will be discussed in Sec. V in the present context.

III. PARTIAL GAUGE GENERATION BY $T(2)$ IN MASSLESS TENSOR GAUGE THEORIES

It was argued, respectively, in [8] and [6,14] that the translational group $T(2)$ in the representation (8) generates gauge transformations in $(3+1)$-dimensional massless KR theory and in linearized gravity. As pointed out in [6], gauge generation by $T(2)$ in linearized gravity is subject to certain restrictions. Here we make a closer examination of this partial gauge generation by $T(2)$ in linearized gravity, revealing some aspects that went unnoticed before. We also show that $T(2)$ generates only a restricted set of gauge transformations in massless KR theory, and that the reducible nature of its gauge transformations is reflected in this partial gauge generation, points that were missed in earlier studies.

We first consider linearized gravity[3] which is governed by the Lagrangian

$$\mathcal{L}_L^E=\frac{1}{2}h_{\mu\nu}\left[R_L^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}R_L\right],$$

$$R_L^{\mu\nu}=\frac{1}{2}(-\Box h^{\mu\nu}+\partial^{\mu}\partial_{\alpha}h^{\alpha\nu}+\partial^{\nu}\partial_{\alpha}h^{\alpha\mu}-\partial^{\mu}\partial^{\nu}h), \quad (11)$$

where $R_L^{\mu\nu}$ is the linearized Ricci tensor while $h=h^{\alpha}_{\alpha}$ and $R_L=R^{\alpha}_{L\alpha}$. Linearized gravity is invariant under the gauge transformation

$$h^{\mu\nu}(x)\rightarrow h'^{\mu\nu}(x)=h^{\mu\nu}(x)+\partial^{\mu}\tilde{\zeta}^{\nu}(x)+\partial^{\nu}\tilde{\zeta}^{\mu}(x). \quad (12)$$

Adopting the ansatz [analogous to Eq. (2)] $h^{\mu\nu}=\chi^{\mu\nu}(k)e^{ik\cdot x}$, where $\chi^{\mu\nu}$ is the symmetric polarization tensor, the gauge transformation (12) can be written in the momentum space as

$$\chi^{\mu\nu}(k)\rightarrow\chi'^{\mu\nu}(k)=\chi^{\mu\nu}(k)+k^{\mu}\zeta^{\nu}(k)+k^{\nu}\zeta^{\mu}(k) \quad (13)$$

with $\tilde{\zeta}^{\mu}(x)=\zeta^{\mu}(k)e^{ik\cdot x}$. Now, following the plane wave method as described in [14], one can obtain the maximally reduced form of the polarization tensor corresponding to a particle with the four-momentum $k^{\mu}=(\omega,0,0,\omega)^T$, given by

$$\{\chi^{\mu\nu}\}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

(For another derivation see [20].) Here a and b are free parameters representing the two physical degrees of freedom in $(3+1)$-dimensional linearized gravity.[4] Notice that the maximally reduced form (14) of the polarization tensor satisfies the momentum space harmonic gauge condition $k_{\mu}\chi^{\mu}_{\nu}=\frac{1}{2}k_{\nu}\chi^{\mu}_{\mu}$ [20].

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[3] In linearized gravity, the metric $g_{\mu\nu}$ is assumed to be close to the flat background part $\eta_{\mu\nu}$, and one writes $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ with deviation $|h_{\mu\nu}|\ll 1$. Raising and lowering of indices are done by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, respectively.

[4] Linearized gravity in d dimensions has $d(d-3)/2$ degrees of freedom [20].

It is now easy to show that the action of the translational group $W(p,q)$ (8) on the polarization tensor (14) is equivalent to a gauge transformation:

$$\{\chi^{\mu\nu}\}\rightarrow\{\chi'^{\mu\nu}\}=W(p,q)\{\chi^{\mu\nu}\}W^T(p,q)$$
$$=\{\chi^{\mu\nu}\}+\begin{pmatrix}[(p^2-q^2)a+2pqb] & (pa+qb) & (pb-qa) & [(p^2-q^2)a+2pqb]\\ (pa+qb) & 0 & 0 & (pa+qb)\\ (pb-qa) & 0 & 0 & (pb-qa)\\ [(p^2-q^2)a+2pqb] & (pa+qb) & (pb-qa) & [(p^2-q^2)a+2pqb]\end{pmatrix}. \tag{15}$$

The above transformation can be cast in the form of a gauge transformation (13) with the following choice for the arbitrary functions $\zeta^\mu(k)$ [14]:

$$\zeta^1=\frac{pa+qb}{\omega},\quad \zeta^2=\frac{pb-qa}{\omega},$$
$$\zeta^0=\zeta^3=\frac{(p^2-q^2)a+2pqb}{2\omega}. \tag{16}$$

However, since $k^\mu=(\omega,0,0,\omega)^T$, the general gauge transformation for $\{\chi^{\mu\nu}\}$ (14) has the form

$$\{\chi^{\mu\nu}\}\rightarrow\{\chi^{\mu\nu}\}+\{k^\mu\zeta^\nu\}+\{k^\nu\zeta^\mu\}$$
$$=\{\chi^{\mu\nu}\}+\omega\begin{pmatrix}2\zeta^0 & \zeta^1 & \zeta^2 & (\zeta^0+\zeta^3)\\ \zeta^1 & 0 & 0 & \zeta^1\\ \zeta^2 & 0 & 0 & \zeta^2\\ (\zeta^0+\zeta^3) & \zeta^1 & \zeta^2 & 2\zeta^3\end{pmatrix}. \tag{17}$$

Upon comparing the above form of general gauge transformation with the one generated by $W(p,q)$ given in Eq. (15), it becomes clear that the latter is only a special case of the former, as the relations in (16) restrict the number of independent components of the arbitrary vector ζ^μ. Therefore, the translational subgroup $T(2)$ of Wigner's little group for massless particles generates only a subset of the full set of gauge transformations in linearized gravity. In this connection one must notice that the gauge freedom in linearized gravity is represented by the arbitrary vector variable ζ^μ, with four components, while the translational group $T(2)$ has only two parameters. Naturally, in gauge generation by $W(p,q)$ in linearized gravity, only two of the four components of ζ^μ remain independent [as is evident from Eq. (17)] when expressed in terms of the two parameters (p,q), and therefore the gauge generation is only partial. It was noted in [6] that the gauge generation by the little group in linearized gravity is subject to the "Lorentz condition" $k_\mu\zeta^\mu(k)=0$. This can also be seen from the third relation $\zeta^0=\zeta^3$ in Eq. (16), since $k^\mu=(\omega,0,0,\omega)^T$. Thus, our present analysis has unraveled all the constraints behind the partial gauge generation by Wigner's little group in linearized gravity. In contrast, the gauge freedom in free Maxwell theory is represented by a single arbitrary scalar variable $f(k)$ (3) which can be expressed (without any restrictions) in terms of the two parameters of $W(p,q)$ in gauge generation by the little group, as is evident from Eq. (9). Hence the translational subgroup of Wigner's little group generates the full set of gauge transformations in Maxwell theory.

We now consider the gauge transformations generated by the translational group $W(p,q)$ in massless KR theory, which has a second rank antisymmetric tensor as its basic field. The KR theory is described by the Lagrangian

$$\mathcal{L}=\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda},\quad H_{\mu\nu\lambda}=\partial_\mu B_{\nu\lambda}+\partial_\nu B_{\lambda\mu}+\partial_\lambda B_{\mu\nu}, \tag{18}$$

where $B_{\mu\nu}$ is the second rank antisymmetric gauge field $(B_{\mu\nu}=-B_{\nu\mu})$. The KR theory is invariant under the gauge transformation

$$B_{\mu\nu}(x)\rightarrow B'_{\mu\nu}(x)=B_{\mu\nu}(x)+\partial_\mu F_\nu(x)-\partial_\nu F_\mu(x), \tag{19}$$

where $F_\mu(x)$ are arbitrary functions. One can see that under the transformation

$$F_\mu(x)\rightarrow F'_\mu(x)=F_\mu(x)+\partial_\mu\beta(x) \tag{20}$$

[where $\beta(x)$ is an arbitrary scalar function] the gauge transformation (19) remains invariant. In particular, if $F_\mu=\partial_\mu\Lambda$, the gauge transformation vanishes trivially. This is known as the "gauge invariance of gauge transformations" and is a typical property of reducible gauge theories[5] where the generators of gauge transformations are not all independent [15]. Hence there exists some superfluity in the gauge transformation (19). The maximally reduced form of the an-

[5]Notice a crucial difference in the case of linearized gravity, which has the symmetric tensor $h_{\mu\nu}$ as its underlying gauge field. Under a transformation of the type (20), the gauge transformation (12) changes. This shows that, unlike in KR theory, there is no gauge invariance of gauge transformation in linearized gravity, which is not a reducible gauge system.

tisymmetric polarization tensor $\varepsilon^{\mu\nu}$ associated with the two-form potential $B^{\mu\nu}$ $(=\varepsilon^{\mu\nu}e^{ik\cdot x})$ of KR theory was obtained in [8] using the plane wave method as

$$\{\varepsilon^{\mu\nu}\}=\varepsilon^{12}\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}. \qquad (21)$$

Notice that, as in Maxwell and linearized gravity theories, this form of the polarization tensor satisfies the Lorentz condition $k_\mu\varepsilon^{\mu\nu}=0$ corresponding to $\partial_\mu B^{\mu\nu}=0$. In the case of KR theory, the counterpart of the momentum space gauge transformation (13) is given by

$$\varepsilon^{\mu\nu}(k)\rightarrow\varepsilon'^{\mu\nu}(k)=\varepsilon^{\mu\nu}(k)+i[k^\mu f^\nu(k)-k^\nu f^\mu(k)], \qquad (22)$$

where $f_\mu(k)$ are arbitrary and independent functions of k [with $F_\mu(x)=f_\mu(k)e^{ik\cdot x}$]. The transformation of $\{\varepsilon_{\mu\nu}\}$ under the translational group $W(p,q)$ (8) can be written as

$$\{\varepsilon^{\mu\nu}\}\rightarrow\{\varepsilon'^{\mu\nu}\}=W(p,q)\{\varepsilon^{\mu\nu}\}W^T(p,q)$$
$$=\{\varepsilon^{\mu\nu}\}+\varepsilon^{12}\begin{pmatrix}0&-q&p&0\\q&0&0&q\\-p&0&0&-p\\0&-q&p&0\end{pmatrix}. \qquad (23)$$

This can be cast in the form of Eq. (22) with

$$f^1=\frac{-q\varepsilon^{12}}{i\omega},\quad f^2=\frac{p\varepsilon^{12}}{i\omega},\quad f^3=f^0. \qquad (24)$$

As in the case of linearized gravity, on account of the requirement $f^3=f^0$, the gauge transformations generated by the translational group also fail to include the entire set of gauge transformations in KR theory. Analogous to Eq. (17), the general form of gauge transformation (22) in the matrix form is

$$\{\varepsilon^{\mu\nu}\}\rightarrow\{\varepsilon'^{\mu\nu}\}=\{\varepsilon^{\mu\nu}\}+\omega\begin{pmatrix}0&f^1&f^2&f^0-f^3\\-f^1&0&0&-f^1\\-f^2&0&0&-f^2\\f^3-f^0&f^1&f^2&0\end{pmatrix}, \qquad (25)$$

which makes it quite explicit that the transformation (23) does not exhaust (25), but is only a special case of it (where $f^0=f^3$). The transformation (23) is an attempt to generate the gauge equivalence class of the maximally reduced polarization tensor (21) of KR theory using only the two parameters of the translational group $W(p,q)$, while the full gauge freedom of the theory is represented by the arbitrary four-vector variable $f^\mu(k)$. Hence, analogous to the case of linearized gravity, gauge generation by $W(p,q)$ in massless KR theory is only partial. Moreover, here also the arbitrary function $f^\mu(k)$ satisfies the Lorentz condition $k_\mu f^\mu(k)=0$ since $f^0=f^3$ [Eq. (24)].

It is important to notice that in Eq. (24), while the components f^1 and f^2 of f^μ are expressed in terms of the parameters p,q of the translational group $W(p,q)$, the other two components (f^0,f^3) are independent of the parameters (and of the maximally reduced polarization tensor) and are left completely undetermined subject only to the constraint $f^0=f^3$. Thus, in gauge generation by $W(p,q)$ in KR theory, corresponding to any given pair (f^1,f^2) there exists a continuum of allowed choices for f^0 $(=f^3)$ representative of the invariance of gauge transformations (19) under (20). Therefore, the partial gauge generation by $W(p,q)$ in massless KR theory clearly exhibits the reducibility of its gauge transformations. This may be compared to the gauge generation (15) in linearized gravity by $W(p,q)$, where all the components of the arbitrary vector variable ζ^μ are expressed in terms of the parameters (p,q) [see Eq. (16)], hence indicating the absence of any reducibility in the gauge transformation of the theory.

Notice that the transformation (20) is of same form as the gauge transformation in Maxwell theory where $W(p,q)$ acts as the gauge generator. Hence, the gauge transformation (20) of gauge transformations in KR theory may be considered as being generated by the translational group $W(p,q)$. Thus, in KR theory, which is a two-form gauge theory, two independent elements of the translational group $W(p,q)$ are involved in generating gauge transformations, one for the underlying two-form field $B_{\mu\nu}$ and the other for the field F_μ, which corresponds to the gauge freedom of the theory. In gauge generation for massless theories by the translational group $W(p,q)$, we therefore perceive an appealing hierarchical structure starting from the Maxwell (one-form) and KR (two-form) theories; namely, in an *n*-form theory, *n* elements of the translational group $W(p,q)$ are involved in gauge generation. It is expected that this hierarchical structure continues for higher form gauge theories as well.

IV. MASSIVE GAUGE THEORIES

In this section we study the relationship between the translational groups and gauge transformation in gauge theories that are obtained from massive theories through the Stückelberg mechanism. Only $(3+1)$-dimensional theories are considered in this section.

A. Massive vector gauge theory

One can render the $(3+1)$-dimensional Proca theory (which does not possess any gauge symmetry) gauge invariant by the Stückelberg mechanism with the introduction of a new scalar field $\alpha(x)$ as follows:

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{m^2}{2}(A_\mu+\partial_\mu\alpha)(A^\mu+\partial^\mu\alpha). \qquad (26)$$

The Lagrangian remains invariant under the transformations

$$A_\mu(x)\to A'_\mu(x)=A_\mu(x)+\partial_\mu\Lambda(x),$$
$$\alpha(x)\to\alpha'(x)=\alpha(x)-\Lambda(x), \tag{27}$$

where $\Lambda(x)$ is an arbitrary scalar function. The equations of motion for the theory are

$$-\partial^\nu F_{\mu\nu}+m^2(A_\mu+\partial_\mu\alpha)=0, \quad \partial^\mu(A_\mu+\partial_\mu\alpha)=0. \tag{28}$$

One must notice that by operating ∂_μ on the first equation in (28) one yields the second. Hence the latter is a consequence of the former. This implies that the gauge transformation of the α field can be deduced by knowing that of the A^μ field. Similarly to Eq. (2), here we adopt the ansatz $A_\mu(x)=\varepsilon_\mu\exp(ik\cdot x)$ and $\alpha(x)=\tilde{\alpha}(k)\exp(ik\cdot x)$. In terms of the polarization vector $\varepsilon_\mu(k)$, the equations of motion (28) become, respectively,

$$k^\nu(k_\mu\varepsilon_\nu-k_\nu\varepsilon_\mu)+m^2(\varepsilon_\mu+ik_\mu\tilde{\alpha})=0, \quad ik^\nu(\varepsilon_\nu+ik_\nu\tilde{\alpha})=0. \tag{29}$$

For massless excitations $k^2=0$, the second equation in (29) gives the Lorentz condition $k_\nu\varepsilon^\nu=0$ which when substituted in the first gives $\varepsilon_\mu=-ik_\mu\tilde{\alpha}$. Since this is a solution proportional to the four-momentum k_μ, it can be gauged away by an appropriate choice of the gauge. Thus massless excitations are gauge artifacts. For $k^2=M^2$ (massive excitations), the equations of motion (29) become

$$(m^2-M^2)\varepsilon^\mu+k^\mu k_\nu\varepsilon^\nu+im^2k^\mu\tilde{\alpha}=0, \quad \tilde{\alpha}=\frac{ik_\nu\varepsilon^\nu}{M^2}. \tag{30}$$

Substituting the second equation of (30) in the first yields

$$(m^2-M^2)\varepsilon^\mu+k_\nu\varepsilon^\nu k^\mu\left(1-\frac{m^2}{M^2}\right)=0. \tag{31}$$

Now, Eq. (31) can be satisfied only if $M=m$. Therefore, the mass of the excitation is given by m itself, and the rest frame momentum four-vector of the theory can be written as $k^\nu=(m,0,0,0)$. Then, in the rest frame the second equation in (29) gives $\varepsilon_0=-im\tilde{\alpha}$. Therefore, the polarization vector of the $A^\mu(x)$ field in Eq. (26) can be written as $\varepsilon^\mu=(-im\tilde{\alpha},\varepsilon^1,\varepsilon^2,\varepsilon^3)^T$, and a gauge transformation with the choice $\Lambda(x)=\alpha(x)$ yields its maximally reduced form

$$\varepsilon^\mu=(0,\varepsilon^1,\varepsilon^2,\varepsilon^3)^T, \tag{32}$$

where the free components $\varepsilon^1,\varepsilon^2,\varepsilon^3$ represents the three physical degrees of freedom in the theory. One must note that Eq. (32) is of the same form as that of the $B\wedge F$ theory polarization vector [8]. Therefore, just as in the case of $B\wedge F$ theory, the action of the representation $D(p,q,r)$ (10) of $T(3)$ on the polarization vector (32) amounts to a gauge transformation in Stückelberg extended Proca theory:

$$\varepsilon^\mu\to\varepsilon'^\mu=D^\mu{}_\nu(p,q,r)\varepsilon^\nu=\varepsilon^\mu+\frac{i}{m}(p\varepsilon^1+q\varepsilon^2+r\varepsilon^3)k^\mu. \tag{33}$$

The above transformation can be cast in the form of the momentum space gauge transformation

$$\varepsilon^\mu\to\varepsilon^\mu+ik^\mu\lambda(k) \tag{34}$$

[where $\Lambda(x)=\lambda(k)e^{ik\cdot x}$] corresponding to the field $A(x)$, by choosing the field $\Lambda(x)$ such that

$$\lambda(k)=\frac{(p\varepsilon^1+q\varepsilon^2+r\varepsilon^3)}{m}. \tag{35}$$

As mentioned before, it is possible to obtain the gauge transformation property of the α field from that of the A^μ field. Consider the second relation in (30), i.e., $\tilde{\alpha}=ik_\mu\varepsilon^\mu/m^2$, and let ε^μ undergo the gauge transformation (34), which has the effect of making a corresponding transformation in the α field:

$$\tilde{\alpha}\to\tilde{\alpha}'=\frac{ik_\mu(\varepsilon^\mu+ik^\mu\lambda)}{m^2}=\frac{ik_\mu\varepsilon^\mu}{m^2}-\lambda=\tilde{\alpha}-\lambda. \tag{36}$$

Here λ is given by Eq. (35), corresponding to the gauge transformation generated by the translational group $T(3)$ in the $A^\mu(x)$ field. Notice that the above equation (36) corresponds to the second equation in (27). We have thus obtained the gauge transformation generated in the α field by $T(3)$ from that in the $A_\mu(x)$ field. It follows therefore that the α field can be gauged away completely by a suitable gauge fixing condition (unitary gauge), and it does not appear in the physical spectrum of the theory.

Hence it is obvious that the representation $D(p,q,r)$ of $T(3)$ generates gauge transformations in the massive vector gauge theory governed by Eq. (26).

B. Massive symmetric tensor gauge theory

Consider the massive and gauge noninvariant Einstein-Pauli-Fierz theory in $3+1$ dimensions as given by the Lagrangian

$$\mathcal{L}_L^{EPF}=\frac{1}{2}h_{\mu\nu}\left[R_L^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}R_L\right]-\frac{\mu^2}{2}[(h_{\mu\nu})^2-h^2]. \tag{37}$$

Just as the Proca theory (Sec. IV A) can be made gauge invariant by the Stückelberg mechanism, the linearized EPF theory can also be made gauge invariant by introducing an additional vector field A^μ as follows:

$$\mathcal{L}_L^{EPF}=\frac{1}{2}h_{\mu\nu}\left[R_L^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}R_L\right]-\frac{\mu^2}{2}[(h_{\mu\nu}+\partial_\mu A_\nu+\partial_\nu A_\mu)^2-(h+2\partial\cdot A)^2]. \tag{38}$$

The theory described by Eq. (38) is invariant under the gauge transformations

$$h_{\mu\nu}\to h'_{\mu\nu}=h_{\mu\nu}+\partial_\mu\Lambda_\nu+\partial_\nu\Lambda_\mu,$$
$$A_\mu(x)\to A'_\mu(x)=A_\mu(x)-\Lambda_\mu(x). \tag{39}$$

The equations of motion for $h_{\mu\nu}$ and A_μ are given, respectively, by the following equations:

$$-\Box h^{\mu\nu}+\partial^\mu\partial_\alpha h^{\alpha\nu}+\partial^\nu\partial_\alpha h^{\alpha\mu}-\partial^\mu\partial^\nu h + \eta^{\mu\nu}(\Box h-\partial_\alpha\partial_\beta h^{\alpha\beta})-\mu^2[(h^{\mu\nu}+\partial^\mu A^\nu+\partial^\nu A^\mu) - \eta^{\mu\nu}(h+2\partial\cdot A)]=0, \tag{40}$$

$$\Box A^\mu+\partial_\nu h^{\nu\mu}-\partial^\mu h-\partial^\mu(\partial\cdot A)=0. \tag{41}$$

Analogously to the case of massive vector gauge theory discussed before, the equation of motion (41) for A^μ can be obtained from Eq. (40) by applying the operator ∂_ν. Therefore, gauge transformation of A^μ is obtainable by knowing the gauge transformation of the $h^{\mu\nu}$ field via a method similar to the one discussed in Sec. IV A for the case of Stückelberg extended Proca theory. With $h_{\mu\nu}(x)=\chi_{\mu\nu}(k)e^{ik\cdot x}$ and $A_\mu(x)=\varepsilon_\mu(k)e^{ik\cdot x}$, we now employ the plane wave method to obtain the maximally reduced polarization tensor $\chi_{\mu\nu}$ involved in the $h_{\mu\nu}$ field. In the momentum space, the gauge transformations (39) can be written as

$$\chi_{\mu\nu}\to\chi'_{\mu\nu}=\chi_{\mu\nu}+ik_\mu\zeta_\nu+ik_\nu\zeta_\mu, \quad \varepsilon_\mu\to\varepsilon'_\mu=\varepsilon_\mu-\zeta_\mu \tag{42}$$

[where $\Lambda_\mu(x)=\zeta_\mu(k)\exp(ik\cdot x)$], and the equation of motion (40) for $h_{\mu\nu}$ as

$$k^2\chi^{\mu\nu}-k^\mu k_\alpha\chi^{\alpha\nu}-k^\nu k_\alpha\chi^{\alpha\mu}+k^\mu k^\nu\chi + \eta^{\mu\nu}(-k^2\chi+k_\alpha k_\beta\chi^{\alpha\beta})-\mu^2[\chi^{\mu\nu}+ik^\mu\varepsilon^\nu+ik^\nu\varepsilon^\mu - \eta^{\mu\nu}(\chi+2ik_\alpha\varepsilon^\alpha)]=0. \tag{43}$$

On contracting with $\eta_{\mu\nu}$ and considering only massless ($k^2=0$) excitations, Eq. (43) reduces to

$$2k_{\mu\nu}\chi^{\mu\nu}+\mu^2[3(\chi+2ik_\mu\varepsilon^\mu)]=0. \tag{44}$$

The solution of the above equation is $\chi^{\mu\nu}=-i(k^\mu\varepsilon^\nu+k^\nu\varepsilon^\mu)$. Hence it is also the solution of Eq. (43) with $k^2=0$. It is obvious that this solution is a gauge artifact since one can choose the arbitrary vector field $\Lambda_\mu=A_\mu$ so as to make this solution vanish.

Next we consider the massive case ($k^2=M^2, M\neq 0$) and consider the (00) component of the equation of motion (43) which, by straightforward algebra, can be reduced to

$$\chi^1{}_1+\chi^2{}_2+\chi^3{}_3=0. \tag{45}$$

Similarly, the $(0i)$ component of Eq. (43) gives $\chi_{0i}=-iM\varepsilon_i$. Now, the (ij) component of Eq. (43) is given by

$$k^2\chi_{ij}-\eta_{ij}k^2(\chi-\chi^{00})-\mu^2[\chi_{ij}-\eta_{ij}(\chi+2iM\varepsilon^0)]=0. \tag{46}$$

Using Eq. (45), the above equation can be reduced to

$$k^2\chi_{ij}-\mu^2[\chi_{ij}-\eta_{ij}(\chi+2iM\varepsilon^0)]=0. \tag{47}$$

By adding up the three equations obtained by successively setting $i=j=1,2,3$ in Eq. (47), and subsequently using Eq. (45), we arrive at $\chi_{00}=-2iM\varepsilon_0$. On the other hand, when $i\neq j$ Eq. (47) reduces to

$$(\mu^2-M^2)\chi_{ij}=0. \tag{48}$$

At this juncture, notice that only two of the three components $\chi_{ii}, i=1,2,3$, are independent on account of Eq. (45). Also, the χ_{00} and χ_{0i} components can be set equal to zero by choosing the arbitrary field Λ_μ to be A_μ. Therefore, if $\chi_{ij}=0$ (for $i\neq j$) in the above equation (48), the number of independent components of $\chi_{\mu\nu}$ will be only two. Since this is not the case, we can satisfy Eq. (48) only if $\mu^2=M^2$. Thus we see that the parameter μ represents the mass of the physical excitations of the field $h_{\mu\nu}$ and that its polarization tensor is

$$\{\chi_{\mu\nu}\}=\begin{pmatrix} -2i\mu\varepsilon_0 & -i\mu\varepsilon_1 & -i\mu\varepsilon_2 & -i\mu\varepsilon_3 \\ -i\mu\varepsilon_1 & \chi_{11} & \chi_{12} & \chi_{13} \\ -i\mu\varepsilon_2 & \chi_{12} & \chi_{22} & \chi_{23} \\ -i\mu\varepsilon_3 & \chi_{13} & \chi_{23} & \chi_{33}\end{pmatrix},$$

$$\chi_{11}+\chi_{22}+\chi_{33}=0. \tag{49}$$

As mentioned before, by choosing the field Λ_μ to be A_μ and making a gauge transformation, the above form of the polarization tensor can be converted to its maximally reduced form given by

$$\{\chi_{\mu\nu}\}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \chi_{11} & \chi_{12} & \chi_{13} \\ 0 & \chi_{12} & \chi_{22} & \chi_{23} \\ 0 & \chi_{13} & \chi_{23} & \chi_{33}\end{pmatrix}, \quad \chi_{11}+\chi_{22}+\chi_{33}=0. \tag{50}$$

The action of $D(p,q,r)$ on the polarization tensor $\{\chi_{\mu\nu}\}$ (50) is given by

$$\{\chi_{\mu\nu}\}\to\{\chi_{\mu\nu}\}'=D(p,q,r)\{\chi_{\mu\nu}\}D^T(p,q,r)$$

$$=\{\chi_{\mu\nu}\}+\begin{pmatrix} \begin{pmatrix} p(p\chi_{11}+q\chi_{12}+r\chi_{13}) \\ +q(p\chi_{12}+q\chi_{22}+r\chi_{23}) \\ +r(p\chi_{13}+q\chi_{23}+r\chi_{33})\end{pmatrix} & \begin{pmatrix} p\chi_{11}+q\chi_{12} \\ +r\chi_{13}\end{pmatrix} & \begin{pmatrix} p\chi_{12}+q\chi_{22} \\ +r\chi_{23}\end{pmatrix} & \begin{pmatrix} p\chi_{13}+q\chi_{23} \\ +r\chi_{33}\end{pmatrix} \\ p\chi_{11}+q\chi_{12}+r\chi_{13} & 0 & 0 & 0 \\ p\chi_{12}+q\chi_{22}+r\chi_{23} & 0 & 0 & 0 \\ p\chi_{13}+q\chi_{23}+r\chi_{33} & 0 & 0 & 0 \end{pmatrix}. \tag{51}$$

By choosing

$$\zeta_0=\frac{1}{2}(p\zeta_1+q\zeta_2+r\zeta_3),$$

$$\zeta_1=\frac{1}{\mu}(p\chi_{11}+q\chi_{12}+r\chi_{13}),$$

$$\zeta_2=\frac{1}{\mu}(p\chi_{12}+q\chi_{22}+r\chi_{23}),$$

$$\zeta_3=\frac{1}{\mu}(p\chi_{13}+q\chi_{23}+r\chi_{33}), \quad \chi_{11}+\chi_{22}+\chi_{33}=0,$$

it is straightforward to see that Eq. (51) has the form of the gauge transformation of $\chi_{\mu\nu}$ [see Eq. (42)]. Notice that, when one makes the choices for the components ζ_1,ζ_2,ζ_3 in terms of the three parameters p,q,r of the translational group $T(3)$, the component ζ_0 is automatically fixed. Therefore, in the gauge transformation (51) generated by the representation $D(p,q,r)$ of $T(3)$, only the three space components of the arbitrary field ζ_μ remain independent, whereas for the generation of the complete set of gauge transformations (42) all four components of ζ_μ should be independent of one another. Hence, the above gauge transformations (51) generated by the translational group $D(p,q,r)$ do not exhaust the complete set of gauge transformations available to the massive symmetric tensor gauge theory. As mentioned before, the gauge transformation of the A_μ field can be obtained from that of the $h_{\mu\nu}$ field, although the former does not appear in the physical spectrum of the theory.

C. Massive antisymmetric tensor gauge theory

Here we discuss the role of the translational group $T(3)$ as gauge generator in the Stückelberg extended massive KR theory, which is another example of a reducible gauge theory. Although the analysis in this case closely resembles that for Stückelberg extended EPF theory, here the reducibility of the gauge transformation is manifested in the gauge generation by $T(3)$. The Lagrangian of the Stückelberg extended massive KR theory is

$$\mathcal{L}=\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}-\frac{m^2}{4}(B_{\mu\nu}+\partial_\mu A_\nu-\partial_\nu A_\mu)\times(B^{\mu\nu}+\partial^\mu A^\nu-\partial^\nu A^\mu). \tag{52}$$

It can easily be verified that this theory is invariant under the joint gauge transformations

$$B_{\mu\nu}(x)\rightarrow B_{\mu\nu}(x)+\partial_\mu F_\nu(x)-\partial_\nu F_\mu(x),$$

$$A_\mu(x)\rightarrow A_\mu(x)-\Lambda_\mu(x). \tag{53}$$

Here we must notice that the gauge transformation of $B_{\mu\nu}$ is reducible exactly like that in massless KR theory. The equations of motion (corresponding to $B_{\mu\nu}$ and A_μ) following from Eq. (52) are given by

$$\partial_\mu H^{\mu\nu\lambda}+m^2(B^{\nu\lambda}+\partial^\nu A^\lambda-\partial^\lambda A^\nu)=0, \tag{54}$$

$$\partial_\mu(B^{\mu\nu}+\partial^\mu A^\nu-\partial^\nu A^\mu)=0. \tag{55}$$

As in the case of the Stückelberg extended massive theories considered previously in Secs. IV A and IV B, the equation of motion (55) for A^ν can be obtained from Eq. (54) by the application of the operator ∂_λ. Hence, one can easily obtain the gauge transformation of the A^ν field from that of the $B^{\nu\lambda}$ field.

In order to obtain the maximally reduced polarization tensor $\varepsilon^{\mu\nu}(k)$ corresponding to the antisymmetric field $B^{\mu\nu}(x)$, as usual we use the ansatz $B^{\mu\nu}(x)=\varepsilon^{\mu\nu}(k)e^{ik\cdot x}$, $A^\mu(x)=\varepsilon^\mu(k)e^{ik\cdot x}$ and employ the plane wave method. The momentum space gauge transformation of $\varepsilon^{\mu\nu}(k)$ now has the same form as Eq. (22). The equation of motion (54) can be written (in momentum space) as

$$-k^2\varepsilon^{\nu\lambda}-k^\nu k_\mu\varepsilon^{\lambda\mu}-k^\lambda k_\mu\varepsilon^{\mu\nu}+m^2(\varepsilon^{\nu\lambda}+ik^\nu\varepsilon^\lambda-ik^\lambda\varepsilon^\nu)=0. \tag{56}$$

If $k^2=0$ (massless excitations), the above equation reduces to

$$-k^\nu k_\mu\varepsilon^{\lambda\mu}-k^\lambda k_\mu\varepsilon^{\mu\nu}+m^2(\varepsilon^{\nu\lambda}+ik^\nu\varepsilon^\lambda-ik^\lambda\varepsilon^\nu)=0, \tag{57}$$

the most general solution for which is

$$\varepsilon^{\nu\lambda}(k)=C(ik^\nu\varepsilon^\lambda-ik^\lambda\varepsilon^\nu)+D(\epsilon^{\nu\lambda\tau\sigma}k_\tau\varepsilon_\sigma), \tag{58}$$

where C and D are constants to be fixed. Substituting this solution (58) in Eq. (57), we can easily see that $C=-1$ and $D=0$. Therefore, the solution to Eq. (56) corresponding to massless excitations is $\varepsilon^{\nu\lambda}(k)=-ik^\nu\varepsilon^\lambda+ik^\lambda\varepsilon^\nu$. However, such solutions can be gauged away by choosing the arbitrary field $\Lambda^\mu(x)=A^\mu(x)$, which shows that massless excitations are gauge artifacts.

Next we consider the massive case $k^2=M^2(M\neq 0)$ where it is possible to go to the rest frame where $k^\mu=(M,0,0,0)^T$. In the rest frame, the equation of motion (56) reduces to

$$(m^2-M^2)\varepsilon^{\nu\lambda}-M(k^\nu\varepsilon^{\lambda 0}+k^\lambda\varepsilon^{0\nu})+m^2(ik^\nu\varepsilon^\lambda-ik^\lambda\varepsilon^\nu)=0. \tag{59}$$

Note that, since the polarization tensor $\varepsilon^{\nu\lambda}$ is antisymmetric, all its diagonal entries are automatically zero. Considering the components of Eq. (59) for which $(\nu=0,\lambda=i)$, we have $\varepsilon^{i0}=iM\varepsilon^i$. For $(\nu=i,\lambda=j)$ with $i\neq j$, Eq. (59) gives $(m^2-M^2)\varepsilon^{ij}=0$. This leads to two possibilities; either $\varepsilon^{ij}=0$ or $M^2=m^2$. The former possibility can be ruled out by the following reasoning. Since Eq. (52) is the first class version (obtained by a Stückelberg extension mechanism) of massive KR theory possessing three physical degrees of freedom [18], the theory described by Eq. (52) also must inherit the same number of degrees of freedom. However, the ε^{i0} elements can all be made to vanish by the gauge choice $\Lambda_\mu=A_\mu$. Therefore the possibility $\varepsilon^{ij}=0$ leads to a null theory and hence should be discounted. So we have $M^2=m^2$, which is also consistent with the number of degrees of freedom. Finally, analogous to Eq. (50), the maximally reduced form of the polarization tensor corresponding to Eq. (52) is given by

$$\{\varepsilon^{\mu\nu}\}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon^{12} & \varepsilon^{13} \\ 0 & -\varepsilon^{12} & 0 & \varepsilon^{23} \\ 0 & -\varepsilon^{13} & -\varepsilon^{23} & 0 \end{pmatrix}. \tag{60}$$

As in the case of Stückelberg extended EPF theory, the A_μ field disappears from the physical spectrum in this case also. Here it must be emphasized that the maximally reduced polarization tensor of $B\wedge F$ theory also has the same form (60). This is not surprising, since the physical sector of $B\wedge F$ theory is equivalent to massive KR theory, whose first class version is the theory (52) under consideration now [18]. We now study the action of $D(p,q,r)$ on Eq. (60), given by

$$\{\varepsilon^{\mu\nu}\}\rightarrow\{\varepsilon^{\mu\nu}\}'=D(p,q,r)\{\varepsilon^{\mu\nu}\}D^T(p,q,r)=\{\varepsilon^{\mu\nu}\}+\begin{pmatrix} 0 & -q\varepsilon^{12}-r\varepsilon^{13} & p\varepsilon^{12}-r\varepsilon^{23} & p\varepsilon^{13}+q\varepsilon^{23} \\ q\varepsilon^{12}+r\varepsilon^{13} & 0 & 0 & 0 \\ -p\varepsilon^{12}+r\varepsilon^{23} & 0 & 0 & 0 \\ -p\varepsilon^{13}-q\varepsilon^{23} & 0 & 0 & 0 \end{pmatrix}. \tag{61}$$

This can be considered to be the gauge transformation of $\varepsilon^{\nu\lambda}$ (22) if we choose

$$f^1=\frac{1}{m}(q\varepsilon^{12}+r\varepsilon^{13}),\quad f^2=\frac{1}{m}(-p\varepsilon^{12}+r\varepsilon^{23}),$$

$$f^3=\frac{-1}{m}(p\varepsilon^{13}+q\varepsilon^{23}). \tag{62}$$

Note that the component f^0 remains completely undetermined and does not depend at all either on the parameters p,q,r of $T(3)$ or on the maximally reduced polarization tensor of the theory, whereas the other components f^1,f^2,f^3 are determined by these parameters and the elements of the polarization tensor. Hence, $T(3)$ generates the complete set of gauge transformations in the Stückelberg extended massive KR theory. Interestingly, it is exactly in the same fashion as in the present case (of Stückelberg extended massive KR theory) that gauge transformations of $B\wedge F$ theory are generated by the translational group $D(p,q,r)$ (we refer to [8] for details). Analogously to the gauge transformation generated by $W(p,q)$ in massless KR theory, for any given set of (f^1,f^2,f^3) we have a continuum of values for f^0, representing the reducibility of the gauge transformation in the underlying two-form field both in the Stückelberg extended first class version of massive KR theory and in the $B\wedge F$ theory. Therefore, the complete independence of the time component of f^μ of the maximally reduced polarization tensor and of the parameters of the group $D(p,q,r)$ is a consequence of the reducibility of the gauge transformations of these theories.

Analogous to the hierarchical structure involving the elements of $T(2)$ present in the gauge generation in massless n-form theories, there is a hierarchical structure in the gauge transformations generated by $T(3)$ in massive n-form theories also. In Sec. IV A we saw that an element of $T(3)$ generates gauge transformations in a massive one-form theory (the Stückelberg extended Proca theory). In the massive two-form (Stückelberg extended massive KR) theory, two elements of $T(3)$ are involved as generators of gauge transformations, one element for the gauge transformation of the field $B^{\mu\nu}$ and a second element for the "gauge transformation" $F^\mu\rightarrow F^\mu+\partial^\mu\beta$, which corresponds to the reducibility of gauge freedom in $B^{\mu\nu}$. [This transformation is of the same form as the first transformation in Eq. (27) corresponding to massive vector theory, and hence may be considered to be generated by $T(3)$.]

V. DIMENSIONAL DESCENT

Dimensional descent [10] is a method by which one can obtain the energy-momentum vector, polarization tensor, and gauge generating representation of the translational subgroup of Wigner's little group, etc., in a massive gauge theory living in a certain space-time dimension from similar results for gauge theories in one higher dimension. In this sense, dimensional descent is a unification scheme for the results presented in the previous sections. A closely related concept is the idea of "dimensional reduction" by which massless theories in a given dimension can be related to massive theories in a space of one lower dimension, as can be seen from [21]. Similar ideas are also used in the context of string theory, where a massive particle is viewed as a massless particle in one higher dimension with the mass being considered as the momentum component along the additional dimension [22].

We begin our discussion of dimensional descent by noting that the translational group $T(3)$, which generates gauge transformations in $(3+1)$-dimensional $B\wedge F$ theory and in the massive excitations of Stückelberg extended Proca and EPF theories, is an invariant subgroup of $E(3)$. Now, just as $E(2)$ is the generator of gauge transformations in four-dimensional Maxwell theory, $E(3)$ generates gauge transformations in five-dimensional Maxwell theory. This indicates that the generators of gauge transformations in the above mentioned massive gauge theories and five-dimensional Maxwell theory are related.

An element of Wigner's little group in five dimensions [10] can be written as

$$W_5(p,q,r;\psi,\phi,\eta)=\begin{pmatrix} 1+\frac{p^2+q^2+r^2}{2} & p & q & r & -\frac{p^2+q^2+r^2}{2} \\ p & & & & -p \\ q & & R(\psi,\phi,\eta) & & -q \\ r & & & & -r \\ \frac{p^2+q^2+r^2}{2} & p & q & r & 1-\frac{p^2+q^2+r^2}{2} \end{pmatrix}, \tag{63}$$

where p,q,r are any real numbers, while $R(\psi,\phi,\eta)\in SO(3)$, with (ψ,ϕ,η) being a triplet of Euler angles. The corresponding element of the translational group $T(3)$ can be trivially obtained by setting $R(\psi,\phi,\eta)$ to be the identity matrix and will be denoted by $W(p,q,r)=W_5(p,q,r;0)$.

Let us now consider free Maxwell theory in five dimensions ($\mathcal{L}=-\frac{1}{4}F^{ij}F_{ij}$, $i,j=0,1,2,3,4$). For a photon of energy ω (in five-dimensional space-time) propagating in the $i=4$ direction, the momentum five-vector is given by

$$k^i=(\omega,0,0,0,\omega)^T. \tag{64}$$

By following the plane wave method and proceeding exactly as in Sec. II, one can show that the maximally reduced form of the polarization vector of the photon is

$$\varepsilon^i=(0,\varepsilon^1,\varepsilon^2,\varepsilon^3,0)^T, \tag{65}$$

where $\varepsilon^1,\varepsilon^2,\varepsilon^3$ represent the three transverse degrees of freedom (since the polarization vector satisfies the "Lorentz gauge" $\varepsilon^i k_i=0$). If we now suppress the last row of the column matrices k^i (64) and ε^i (65), we end up, respectively, with the energy-momentum four-vector and the polarization vector of the Stückelberg extended Proca model in $3+1$ dimensions. This is equivalent to applying the projection operator given by the matrix

$$\mathcal{P}=\mathrm{diag}(1,1,1,1,0) \tag{66}$$

to the momentum five-vector (64) and the polarization vector (65). Similarly, it is possible to derive the polarization tensor of Stückelberg extended EPF theory (38) from that of linearized Einstein gravity in five dimensions by a procedure analogous to the one described above. As was done in the $(3+1)$-dimensional case, one can easily show that the maximally reduced form of the polarization tensor of $(4+1)$-dimensional linearized gravity is

$$\{\chi^{ij}\}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \chi^{11} & \chi^{12} & \chi^{13} & 0 \\ 0 & \chi^{12} & \chi^{22} & \chi^{23} & 0 \\ 0 & \chi^{13} & \chi^{23} & \chi^{33} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \chi^{11}+\chi^{22}+\chi^{33}=0. \tag{67}$$

By suppressing the last row and column of the polarization tensor (67), one obtains the polarization tensor (50) of the Stückelberg extended EPF model in $3+1$ dimensions. For this purpose, consider the action of $W(p,q,r)$ on ε^i:

$$\begin{aligned} \varepsilon^i\rightarrow\varepsilon'^i&=\varepsilon^i+\delta\varepsilon^i=W_5(p,q,r)^i{}_j\varepsilon^j \\ &=\varepsilon^i+(p\varepsilon^1+q\varepsilon^2+r\varepsilon^3)\frac{k^i}{\omega}. \end{aligned} \tag{68}$$

This is indeed a gauge transformation in $(4+1)$-dimensional Maxwell theory. Applying the projection operator $\mathcal{P}$ (66) on (68) yields

$$\delta\varepsilon^\mu=\mathcal{P}\delta\varepsilon^i=\frac{1}{\omega}(p\varepsilon^1+q\varepsilon^2+r\varepsilon^3)k^\mu. \tag{69}$$

Here $\varepsilon^\mu=(0,\varepsilon^1,\varepsilon^2,\varepsilon^3)^T$ corresponds to the polarization vector in the Stückelberg extended Proca theory and k^μ is the momentum vector of a particle at rest in $3+1$ dimensions. (Here, the time component ω of a five-dimensional massless particle, moving along the extra fifth dimension, is identified with the mass ω of a massive particle at rest in four-dimensional space-time.) Modulo an *i* factor, this is precisely how the polarization vectors in the massive gauge theory (26) transform under gauge transformations (see Sec. IV A). The form of Eq. (69) makes it obvious that

$$\delta\varepsilon^\mu=D(p,q,r)\varepsilon^\mu-\varepsilon^\mu, \tag{70}$$

where $D(p,q,r)$ is given by Eq. (10). Thus, in this fashion we are able to derive the gauge generating representation $D(p,q,r)$ of $T(3)$ in a massive vector field by a judicious application of the projection operator $\mathcal{P}$ (66) from the gauge transformation relation of a higher dimensional massless gauge theory.

We now consider the polarization matrix $\{\chi^{ij}\}$ (67) of linearized gravity in five dimensions,[6] for which the transformation under the action of $W(p,q,r)$ is given by

[6]In this regard, we recollect a comment made in [14] that a translational subgroup of Wigner's little group for massless particles generates gauge transformations only in the $(3+1)$-dimensional version of linearized gravity, but not in its higher dimensional versions. This was mistakenly ascribed to the mismatch in the number of degrees of freedom $[d(d-3)/2]$ in higher dimensional linearized gravity and the number of parameters $(d-2)$ of the translational subgroup of Wigner's little group for massless particles in $d>4$. However, this needs to be amended, as this mismatch is of no consequence in this regard, and it must be stated that the translational subgroup generates gauge transformations for linearized gravity in any dimension $d\geqslant 4$.

$$\{\chi^{ij}\}\rightarrow\{\chi'^{ij}\}=W(p,q,r)\{\chi^{ij}\}W^T(p,q,r)=\{\chi^{ij}\}+\delta\{\chi^{ij}\},$$

where

$$\delta\{\chi^{ij}\}=\{\delta\chi^{ij}\}$$
$$=\begin{pmatrix} ap+bq+rc & a & b & c & ap+bq+rc \\ a & 0 & 0 & 0 & a \\ b & 0 & 0 & 0 & b \\ c & 0 & 0 & 0 & c \\ ap+bq+rc & a & b & c & ap+bq+rc \end{pmatrix} \qquad (71)$$

with $a=p\chi^{11}+q\chi^{12}+r\chi^{13}$, $b=p\chi^{12}+q\chi^{22}+r\varepsilon^{23}$, and $c=p\chi^{13}+q\chi^{23}+r\chi^{33}$ with $\chi^{11}+\chi^{22}+\chi^{33}=0$. Again this can easily be recognized as a gauge transformation[7] in $(4+1)$-dimensional linearized gravity involving massless quanta, as $\delta\chi^{ij}$ can be expressed as $[k^i\zeta^j(k)+k^j\zeta^i(k)]$ with a suitable choice for $\zeta^i(k)$, where $k^i=(\omega,0,0,0,\omega)^T$. By applying the projection operator $\mathcal{P}$ on Eq. (71), we get the change (under gauge transformation) in the $(3+1)$-dimensional polarization matrix $\{\chi^{\mu\nu}\}$ (of Stückelberg extended EPF theory), by the formula $\delta\{\chi^{\mu\nu}\}=\mathcal{P}\delta\{\chi^{ij}\}\mathcal{P}^T$. This simply amounts to a deletion of the last row and column of $\delta\{\chi^{ij}\}$. The result can be expressed more compactly as

$$\delta\{\chi^{\mu\nu}\}=(D\{\chi^{\mu\nu}\}D^T-\{\chi^{\mu\nu}\}), \qquad (72)$$

where $D=D(p,q,r)$ [see Eq. (10)]. This has the precise form of the gauge transformation of the polarization matrix of the Stückelberg extended EPF model, since it can be cast in the form

$$\delta\chi^{\mu\nu}=[k^\mu\zeta^\nu(k)+k^\nu\zeta^\mu(k)] \qquad (73)$$

for a suitable $\zeta^\mu(k)$, where $k^\mu=(\mu,0,0,0)^T$. Here we have identified ω with μ.

Clearly, the generators $T_1=\partial D/\partial p, T_2=\partial D/\partial q, T_3=\partial D/\partial r$ provide a commuting Lie algebra basis for the group $T(3)$. One can easily verify that

$$D(p,q,r)=e^{pT_1+qT_2+rT_3}=1+pT_1+qT_2+rT_3, \qquad (74)$$

so that the change in the polarization vector ε^μ can be expressed as the action of a Lie algebra element

$$\delta\varepsilon^\mu=(pT_1+qT_2+rT_3)\varepsilon^\mu. \qquad (75)$$

In addition, $D(p,q,r)$ also preserves the four-momentum of a particle at rest. Thus we have shown how this representation (10) of $T(3)$ can be connected to Wigner's little group for massless particles in $4+1$ dimensions through appropriate projection in the intermediate steps, where the massless particles moving in $4+1$ dimensions can be associated with a massive particle at rest in $3+1$ dimensions.

[7]Exactly as in the four-dimensional case, this gauge transformation forms only a subset of the full set of gauge transformation available in five-dimensional linearized gravity.

The method of dimensional descent as applied in the case of $(3+1)$-dimensional $B\wedge F$ theory was earlier discussed in [10], where it was shown that one can arrive at the representation $D(p,q,r)$ of the translational group $T(3)$ by considering the gauge transformation properties of $(4+1)$-dimensional massless KR theory. Since the physical sectors of Stückelberg extended massive KR theory and $B\wedge F$ theory are equivalent (and hence possess identical rest frame momentum four-vectors and maximally reduced polarization tensors), it is possible to obtain the gauge generating representation $D(p,q,r)$ of $T(3)$ for Stückelberg extended massive KR theory using dimensional descent, proceeding exactly as was done in [10] for the case of $B\wedge F$ theory.

VI. CONCLUSION

The results of this study can be summarized as follows. We have shown that the representation of the translational $T(3)$ that acts as a gauge generator in the topologically massive $B\wedge F$ gauge theory also generates gauge transformations in the Stückelberg extended gauge invariant versions of the $(3+1)$-dimensional Proca, Einstein-Pauli-Fierz, and massive Kalb-Ramond theories. This representation of $T(3)$ along with the polarization vectors and tensors and the momentum four-vectors of these Stückelberg extended theories are derived systematically using the method of dimensional descent by starting from the appropriate massless gauge theories living in $(4+1)$-dimensional space-time. We have also reexamined the gauge generation in $(3+1)$-dimensional massless tensor gauge theories (linearized gravity and Kalb-Ramond theories) by the translational group $T(2)$ and showed that gauge transformations generated by $T(2)$ in these theories form only a subset of the whole spectrum of gauge transformations available. Similarly, the gauge generation by $T(3)$ in the Stückelberg extended Einstein-Pauli-Fierz theory is also partial. However, in the $B\wedge F$ and Stückelberg extended massive Kalb-Ramond theories, the full set of gauge transformations is generated by $T(3)$. In this connection, we have clarified several subtle points concerning gauge generation by translational groups. It should be emphasized that in the case of reducible gauge systems (massless Kalb-Ramond theory, $B\wedge F$ theory, and the Stückelberg extended massive Kalb-Ramond theory) gauge generation by the relevant translational groups manifestly exhibits the reducibility of the gauge transformations. Furthermore, a hierarchical structure is noticed in the gauge generation by translational groups in both massive and massless theories having *n*-form fields as their basic gauge fields, namely, *n* independent elements of the corresponding translational group are involved in the gauge generation in an *n*-form gauge theory.

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